Physics Placement Exam: Classical Mechanics and Electromagnetism

27-Aug-24

Problem 1: Consider the motion of a point particle of mass m subject to the central potential

$$V(r) = -\frac{\alpha}{r} - k \log r \,, \qquad \alpha > 0 \,, \quad k > 0 \,. \tag{1}$$

(a) Show that there is a critical value L_c for the angular momentum, above which there are no circular orbits. Compute L_c.

(b) How many circular orbits are there for $L < L_c$?

(c) Sketch a plot of the effective potential, for $L > L_c$ and for $L < L_c$.

(d) In the two cases, discuss qualitatively the possible orbits as a function of their energy E.
"Discuss qualitatively" means: characterize the orbits as bound or unbound — no need to provide more details or explicit formulas; just refer to plot features.

Problem 2: Consider two particles on a line with Lagrangian

$$L = \tfrac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \left(e^{x_2} + e^{-x_1} + e^{x_1 - x_2}\right) . \tag{2}$$

The potential energy is minimized at $x_1 = x_2 = 0$. By expanding L to quadratic order in x, find the frequencies of small oscillations around this minimum.

Problem 3: Let x be the position of a particle on a line and p its canonically conjugate momentum, so the Poisson bracket of two functions A, B on phase space is $\{A, B\} = \partial_x A \,\partial_p B - \partial_p A \,\partial_x B$, and the Hamilton equations imply $\dot{A} = \{A, H\}$. Consider the Hamiltonian

$$H \equiv \frac{p^2}{2} + \frac{\lambda}{2x^2} \,. \tag{3}$$

and define moreover

$$D \equiv xp \,, \qquad K \equiv \frac{x^2}{2} \,. \tag{4}$$

(a) Check that the Poisson brackets of H, D and K take the form

$$\{D, H\} = c_1 H \,, \qquad \{K, H\} = c_2 D \,, \qquad \{K, D\} = c_3 K \tag{5}$$

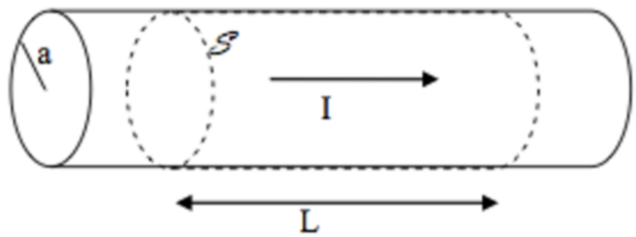
where c_1, c_2, c_3 are λ-independent constants. Find these constants.

(b) Using the relations (5) and $\dot{A} = \{A, H\}$, find $H(t)$, $D(t)$, and $K(t)$ in terms of their $t = 0$ initial values H_0, D_0 and K_0. Then use this together with the definitions (4) to read off the solutions for $x(t)$ and $p(t)$ in terms of x_0 and p_0.

Problem 4: The potential on the surface of a sphere (radius R) is given by
$V(r = R) = V_0 \cos 3\theta$.
a) Find the potential inside and outside the sphere.
b) What is the total charge of the sphere?

Hint: Use trigonometric identities to express the potential entirely in powers of $\cos\theta$

Problem 5:



Consider an infinite wire with cross sectional radius a and conductivity σ. A current I flows down the wire in the $\hat{z}$ direction. The electric field inside the wire is $\vec{E} = \left(\frac{I}{\pi a^2 \sigma}\right) \hat{z}$ and points along the direction of the current, while the magnetic field at the surface of the wire is $\vec{B} = \frac{\mu_0 I}{2\pi a}\hat{\phi}$ (here $\hat{\phi}$ wraps around the wire according to the right hand rule along the direction of I).

a) What is the magnitude and direction of the Poynting flux $\vec{S}$ at the surface of the wire?

b) Consider an imaginary closed cylindrical surface $\mathcal{S}$ that just encloses a wire (see above figure). The integral of the Poynting vector $\vec{S}$ over this surface is given by $\oint \vec{S} \cdot d\vec{A}$. What are the units of this integral? (given the simplest possible units) And what does this surface integral represent physically?

c) For the surface $\mathcal{S}$, it is possible to work out the value of the surface integral $\oint \vec{S} \cdot d\vec{A}$ in two different ways: (1) by computing it directly from $\vec{E}$ and $\vec{B}$; or (2) by knowing what the integral represents physically and simply writing down an expression for that thing.
Work out the value of $\oint \vec{S} \cdot d\vec{A}$ using <u>one</u> of these methods, and state which method you are using.
Your final answer must be in terms of just those variables given in the problem.

Problem 6: A point charge q, of mass m, is attached to a spring of constant k. At time $t = 0$ it is given a kick, so its initial energy is $U_0 = \frac{1}{2} m v_0^2$. Now it oscillates, gradually radiating away this energy. Assume the radiation damping is small, so you can write the equation of motion as

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$$

and the solution as

$$x(t) = \frac{v_0}{\omega_0} e^{-\gamma t/2} \sin \omega_0 t \quad ,$$

with $\omega_0 = \sqrt{\frac{k}{m}}$, $\gamma = \omega_0^2 \tau$ and $\gamma \ll \omega_0$ (drop γ^2 in comparison to ω_0^2, and when you average over a complete cycle, ignore the change in $e^{-\gamma t}$).

a) Determine the value of τ in terms of the other parameters in the problem.

b) Confirm that the total energy radiated is equal to U_0.

Potentially Useful Equations and Definitions

Trigonometric identities: $\cos(A+B) = \cos A \cos B - \sin A \sin B \quad , \quad \sin(A+B) = \sin A \cos B + \sin B \cos A$

Cylindrical coordinates: $x = s\cos\phi$, $y = s\sin\phi$, $z = z$, $\quad s = \sqrt{x^2+y^2}$

$$
\begin{aligned}
\nabla t &= \frac{\partial t}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\,\hat{\phi} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}} \\
\nabla \cdot \mathbf{v} &= \frac{1}{s}\frac{\partial (s v_s)}{\partial s} + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \\
\nabla \times \mathbf{v} &= \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial (s v_\phi)}{\partial s} - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}} \\
\nabla^2 t &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}
\end{aligned}
$$

Spherical coordinates: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$$
\begin{aligned}
\nabla t &= \frac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\,\hat{\phi} \\
\nabla \cdot \mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \\
\nabla \times \mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta\, v_\phi)}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial (r v_\phi)}{\partial r}\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi} \\
\nabla^2 t &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}
\end{aligned}
$$

Solutions when there is no ϕ dependence:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l\, r^l + \frac{B_l}{r^{l+1}}\right] P_l(\cos\theta) \quad \text{satisfies} \quad \nabla^2\Phi = 0 \quad .$$

$$P_0(u) = 1\ ,\ P_1(u) = u\ ,\ P_2(u) = \frac{3}{2}u^2 - \frac{1}{2}\ ,\ P_3(u) = \frac{5}{2}u^3 - \frac{3}{2}u$$

$$\int_{-1}^{1} P_m(u)P_n(u)\,du = \frac{2}{2n+1}\,\delta_{m,n}$$

Electrostatic energy (W):

Discrete charges:

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n}\sum_{j\neq i}^{n} \frac{q_i q_j}{r_{ij}}$$

Continuous charge distribution:

$$W = \frac{\epsilon_0}{2} \int \mathbf{E}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})\, d^3r = \frac{1}{2} \int \rho(\mathbf{r})\Phi(\mathbf{r})\, d^3r$$

Field from magnetic dipole:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{3m} \cdot \hat{\mathbf{r}} - \mathbf{m}}{r^3}$$

Energy density and flux, momentum density:

$$u = \frac{1}{2}\epsilon_0\, \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0}\mathbf{B} \cdot \mathbf{B} \quad , \quad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \quad , \quad \mathbf{g} = \epsilon_0(\mathbf{E} \times \mathbf{B})$$

Speed of light, impedance of the vacuum:

$$\mu_0\epsilon_0 = \frac{1}{c^2} \quad , \quad \mu_0 c = 377\ \Omega$$

Ohm's Law: $\vec{J} = \sigma\vec{E}$

Larmor formula:

$$P = \frac{2}{3} \frac{q^2}{4\pi\epsilon_0 c^3}\, a^2$$